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THE LEVY–STEINITZ REARRANGEMENT THEOREM FOR DUALS OF METRIZABLE SPACES

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ABSTRACT

Extending the Levy–Steinitz rearrangement theorem in $\mathbb{R}^n$, which in turn extended Riemann's theorem, Banaszczyk proved in 1990/93 that a metrizable, locally convex space is nuclear if and only if the domain of sums of every convergent series (i.e. the set of all elements in the space which are sums of a convergent rearrangement of the series) is a translate of a closed subspace of a special form. In this paper we present an apparently complete analysis of the domains of sums of convergent series in duals of metrizable spaces or, more generally, in (DF)-spaces in the sense of Grothendieck.

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Introduction

For a convergent series $\sum(u_k)$ in a locally convex space E the domain of sums $S\big(\sum(u_k)\big)$ is the set of all $x \in E$ which can be obtained as the sum of a convergent rearrangement of $\sum(u_k)$. In terms of this notion Riemann's famous rearrangement theorem states that in the real line $\mathbb{R}$ the domains of sums are either single points or coincide with the whole line. Later Levy [L] and Steinitz [S] extended Riemann's result to finite dimensional spaces: domains of sums in $\mathbb{R}^n$ are affine subspaces — more precisely, for each convergent series $\sum(u_k)$ in $\mathbb{R}^n$

$$S\big(\sum(u_k)\big) = \sum_{k=1}^{\infty} u_k + \Big\{ y \in \mathbb{R}^n \,\big|\, \langle x, y\rangle = 0 \text{ for all } x \in \mathbb{R}^n \text{ with } \sum_{k=1}^{\infty} |\langle x, u_k\rangle| < \infty \Big\};$$

we refer to this result as the "Levy–Steinitz theorem" (see [Ro] and [KK] for a proof). For the "state of art" of this theorem in infinite dimensions (in particular, in Banach spaces) see the recent monograph [KK] of M.I. Kadets and V.M. Kadets.

The following notation is basic to the understanding of domains of sums in infinite dimensional spaces: for each convergent series $\sum(u_k)$ in a locally convex space E define the set

$$\Gamma\big(\sum(u_k)\big) := \big\{ x' \in E' \,\big|\, \big(x'(u_k)\big) \in \ell_1 \big\} \subset E'$$

and its polar

$$\Gamma^{\perp}\big(\sum(u_k)\big) := \big\{ x \in E \,\big|\, x'(x) = 0 \text{ for all } x' \in \Gamma\big(\sum(u_k)\big) \big\} \subset E.$$

Obviously, both sets are subspaces of E' and E, respectively, and the second one is even closed. It is an easy exercise to check the following remark which is useful later (for $E = \mathbb{R}^n$ see also [CC, Thm. 1]):

REMARK: *Let E be a locally convex space and $\sum(u_k)$ a convergent series. Then for each $x \in E$ the following are equivalent:*

(1) $x \in \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}\big(\sum(u_k)\big)$,

(2) $\forall x' \in E'$ $\exists$ *permutation* π *of* $\mathbb{N}$: $x'(x) = \sum_{k=1}^{\infty} x'(u_{\pi(k)})$.

In two papers [B1], [B2] from 1990 and 1993 Banaszczyk proved the following extension of the Levy–Steinitz theorem — a result which here will be quoted as "Banaszczyk's rearrangement theorem":

> *A metrizable, locally convex space E is nuclear if and only if the domain of sums of each convergent series $\sum(u_k)$ in E is given by the formula*
>
> $$S\big(\sum(u_k)\big) = \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}\big(\sum(u_k)\big);$$

in particular, $S\big(\sum(u_k)\big)$ *is a closed, affine subspace of* E*, and for each* $x \in E$*:*

$$x \in S\big(\sum(u_k)\big) \text{ iff } \forall x' \in E' : x'(x) \in S\Big(\sum\big(x'(u_k)\big)\Big).$$

The aim of this paper is to investigate the domains of sums in duals of metrizable spaces or, more generally, in (DF)-spaces in the sense of Grothendieck (in fact, in [KK, p. 101] the authors write that "Steinitz-type problems for locally convex spaces are poorly investigated"). Almost all natural spaces of functions and distributions are metrizable spaces, or duals of metrizable spaces or, if not of this type, then at least "generated" by spaces of this type. The only infinite dimensional (DF)-space for which the Levy–Steinitz theorem was known to hold is the countable direct sum $\varphi := \bigoplus_{\mathbb{N}} \mathbb{R}$ of copies of the scalar field $\mathbb{R}$ (this space is the strong dual of the nuclear Fréchet space $w := \prod_{\mathbb{N}} \mathbb{R}$); we refer to [KK, Ex. 8.3.3]. Implementing the basic ingredients of [B1] and [B2] in an alternative way into locally convex spaces, we are able to say precisely which parts of Banaszczyk's theorem transfer to duals of metrizable spaces and which don't.

Let us briefly describe the content of this article. After a short sketch of Banaszczyk's theorem in section 1, we show in section 2 the following rearrangement theorem: if E is the dual of a nuclear, metrizable space, then each domain of sums $S\big(\sum(u_k)\big)$ is the translate of a subspace of E of a special form:

$$S\big(\sum(u_k)\big) = \sum_{k=1}^{\infty} u_k + \Gamma_{\mathrm{loc}}^{\perp}\big(\sum(u_k)\big).$$

If E is not isomorphic to φ, then there is a convergent series whose domain of sums is not closed — in particular, the Levy–Steinitz theorem in its original formulation does not hold in the dual of a nuclear, metrizable space unless this space equals the trivial space φ. In section 3 it is proved conversely that if E is a dual of a nonnuclear metrizable space, then there is a convergent series whose domain of sums does not have this special form.

An analysis of our approach using local convergence and bounded sets permits to show that in a much larger class of (nonmetrizable) spaces the domain of sums of all convergent series are affine subspaces; this class includes, for example: the space $H(K)$ of germs of holomorphic functions on a compact set K, the space of Schwartz tempered distributions S', the space $\mathcal{E}'(\Omega)$ of distributions with compact support, the space $\mathcal{D}(\Omega)$ of test functions for distributions, the space $\mathcal{D}'(\Omega)$ of distributions, the space $A(\Omega)$ of real analytic functions and its dual $A'(\Omega)$, for an open subset Ω of $\mathbb{R}^N$.

Preliminaries

We shall use standard notation and notions from the theory of locally convex spaces and Banach spaces (see e.g. [BPC], [G], [Ja], [Ju] and [MV]); for the theories of operator ideals and s-numbers see [K], [P2] and [P3]. All locally convex spaces E are real, by $cs(E)$ we denote the collection of all continuous seminorms, and by $\mathcal{B}(E)$ all absolutely convex, closed and bounded sets of E. The dual E' endowed with the topology of uniform convergence on all bounded sets is denoted by E'_b. A sequence (p_n) in $cs(E)$ (or (B_n) in $\mathcal{B}(E)$) is said to be a fundamental system of $cs(E)$ (or $\mathcal{B}(E)$) whenever it is increasing (natural order) and for each $p \in cs(E)$ (or $B \in \mathcal{B}(E)$) there is n such that $p \leq p_n$ (or $B \subset B_n$). Recall that a locally convex space is metrizable if and only if it allows a fundamental system (p_n) of $cs(E)$, and a locally convex space E is said to be a (DF)-space if it has a fundamental system (B_n) of $\mathcal{B}(E)$ and, moreover, each intersection of any sequence of closed, absolutely convex zero neighbourhoods in E is again a zero neighbourhood, provided it absorbs all bounded sets. Strong duals of metrizable spaces are (DF). For a seminorm p on a vector space E we write E_p for the normed space $(E/\ker p, \|\cdot\|_p)$, $\|x + \ker p\|_p := p(x)$, and for an absolutely convex set $B \subset E$ the seminormed space $(\operatorname{span} B, m_B)$, m_B the Minkowski gauge functional, is denoted by E_B. The canonical surjection $E \longrightarrow E_p$ and injection $E_B \hookrightarrow E$ are denoted by π_p and i_B, respectively. Clearly, we have $\overset{\circ}{B}_{E_B} \subset B \subset B_{E_B}$ for the open and closed unit ball of E_B. The set B is said to be a Banach or Hilbert disc whenever E_B is a Banach or Hilbert space. For a locally convex space E and $B \in \mathcal{B}(E)$ the natural embedding i_B is continuous, hence $B = B_{E_B}$ (B is closed in E_B). In complete spaces E all $B \in \mathcal{B}(E)$ are Banach discs (see [BPC, 5.1.27] or [MV, 23.14]). A locally convex space E is called nuclear if for each $p \in cs(E)$ there is $p \leq q \in cs(E)$ such that the canonical linking map $\pi^p_q : E_q \longrightarrow E_p$ is nuclear, and E is said to be co-nuclear if its dual is nuclear or, equivalently, for each $B \in \mathcal{B}(E)$ there is $B \subset C \in \mathcal{B}(E)$ such that the embedding $i^C_B : E_B \hookrightarrow E_C$ is nuclear. Metrizable spaces and (DF)-spaces are nuclear if and only if they are co-nuclear, and nuclear Fréchet spaces and complete nuclear (DF)-spaces are in a one-to-one relation with respect to the correspondence $E \longrightarrow E'_b$ (see [Ju, 7.8.2]). For further details on nuclearity we refer to [Ju], [MV] or [P1]. A sequence (x_n) in a locally convex space E is said to converge locally whenever it is contained and converges in some E_B, $B \in \mathcal{B}(E)$. A locally convex space E satisfies the strict Mackey condition (see e.g. [BPC, 5.1.29]) if given any $B \in \mathcal{B}(E)$ there is $C \in \mathcal{B}(E)$, $C \supset B$, such that E and E_C induce the same topology on B. Clearly, in every locally convex space with

the strict Mackey condition every convergent sequence converges locally. Every metrizable space satisfies the strict Mackey condition ([BPC, 5.1.30]) as well as every complete nuclear (DF)-space (use e.g. [Ju, 7.3.7]).

1. On Banaszczyk's rearrangement theorem for nuclear Fréchet spaces

For later use (and in order to set up some further notation) we want to give a short sketch of the proof of Banaszczyk's rearrangement theorem (which still follows the line of the original proof of Steinitz for $\mathbb{R}^n$).

Let $\sum(u_k)$ be a convergent series in a locally convex space E. The extended domain of sums $S^e\big(\sum(u_k)\big)$ is defined as the set of all $x \in E$ which appear as limits of convergent subsequences of rearrangements $\sum\big(u_{\pi(k)}\big)$. Moreover, let

$$A\big(\sum(u_k)\big) := \bigcap_m Z_m\big(\sum(u_k)\big), \quad Q\big(\sum(u_k)\big) := \bigcap_m co\Big(Z_m\big(\sum(u_k)\big)\Big),$$

where

$$Z_m\big(\sum(u_k)\big) := \Big\{\sum_I u_k \;\Big|\; I \subset \{m, m+1, \ldots\} \text{ finite } \Big\},$$

and

$$A_E\big(\sum(u_k)\big) := \bigcap_m \overline{Z}_m^E \text{ and } Q_E\big(\sum(u_k)\big) := \bigcap_m \overline{co\, Z_m}^E.$$

The proof of part (a) in the following result is elementary (for a more general formulation for abelian Hausdorff groups see [Bo, Ch. III, §5, Ex. 3]), and (b) is a consequence of an easy Hahn-Banach argument (check the proof of [B1, Lemma 6] or see [Bl]).

LEMMA 1.1: *Let $\sum(u_k)$ be a convergent series in a locally convex space E. Then*
(a) $S^e\big(\sum(u_k)\big) = \sum_{k=1}^{\infty} u_k + A_E\big(\sum(u_k)\big)$, *whenever E is metrizable;*
(b) $Q_E\big(\sum(u_k)\big) = \Gamma^{\perp}\big(\sum(u_k)\big)$.

In the metrizable case this obviously gives the following chain of equalities and inclusions:

LEMMA 1.2: *Let $\sum(u_k)$ be a convergent series in a metrizable space E. Then*

$$\text{(B1)} \qquad S\big(\sum(u_k)\big) \subset S^e\big(\sum(u_k)\big) = \sum_{k=1}^{\infty} u_k + A_E\big(\sum(u_k)\big)$$

$$\text{(B2)} \qquad \subset \sum_{k=1}^{\infty} u_k + Q_E\big(\sum(u_k)\big)$$

$$= \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}\Big(\sum (u_k)\Big).$$

In nuclear metrizable spaces the inclusions (B1) and (B2) are even equalities — these are the crucial steps in Banaszczyk's paper [B1]. The proofs are consequences of the following two lemmas (which are variants of [B1, Lemma 4 and Lemma 8]). Here $HS(T\colon X \longrightarrow Y)$ means the Hilbert–Schmidt norm of an operator T acting between the Hilbert spaces X and Y.

LEMMA 1.3 (the "lemma of rounding off coefficients"): *Let H_1 and H_2 be two Hilbert spaces such that $H_1 \subset H_2$ and*

$$HS(\mathrm{id}: H_1 \hookrightarrow H_2) \leq 1.$$

Then for $y_1, \dots, y_s \in B_{H_1}$ and $y \in co\Big\{\sum_J y_k \mid J \subset \{1, \dots, s\}\Big\}$ there is a finite set $I \subset \{1, \dots, s\}$ such that

$$\sum_I y_k - y \in B_{H_2}.$$

LEMMA 1.4 (the "permutation lemma"): *Let H_k, $k = 1, 2, 3$, be three Hilbert spaces such that $H_1 \subset H_2 \subset H_3$ and*

$$HS(\mathrm{id}: H_1 \hookrightarrow H_2) \leq 1, \quad HS(\mathrm{id}: H_2 \hookrightarrow H_3) \leq 1/2.$$

Then for $v_1, \dots, v_s \in B_{H_1}$ and $a \in B_{H_2}$ with $a + \sum_{k=1}^{s} v_k \in B_{H_2}$ there is a permutation σ of $\{1, \dots, s\}$ such that

$$a + \sum_{k=1}^{m} v_{\sigma(k)} \in B_{H_3} \quad \text{for all } m = 1, \dots, s.$$

2. The rearrangement theorem for nuclear (DF)-spaces

In this section we will analyze the domain of sums of convergent series in complete nuclear (DF)-spaces E. Clearly, $\mathbb{R}^n$ and $\varphi = \bigoplus_{\mathbb{N}} \mathbb{R}$ are examples of complete nuclear (DF)-spaces for which the Levy–Steinitz theorem holds — and we will show that these are in fact the only ones with this property, although all domains of sums in complete nuclear (DF)-spaces turn out to be affine subspaces.

Take a convergent series $\sum(u_k)$ in a nuclear (DF)-space E, and recall from the preliminaries that it converges locally, i.e., there is some $B \in \mathcal{B}(E)$ such that

$\sum(u_k)$ converges in E_B. Moreover, let (B_n) be a fundamental system of $\mathcal{B}(E)$ such that $B_1 = B$. Then it follows from Lemma 1.2 that, for each n,

$$\begin{aligned} S_{B_n}\big(\sum(u_k)\big) &\subset S^e_{B_n}\big(\sum(u_k)\big) \\ &= \sum_{k=1}^{\infty} u_k + A_{B_n}\big(\sum(u_k)\big) \\ &\subset \sum_{k=1}^{\infty} u_k + Q_{B_n}\big(\sum(u_k)\big) = \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}_{B_n}\big(\sum(u_k)\big), \end{aligned}$$

where the index B_n indicates that we consider all sets involved with respect to the convergent series $\sum(u_k)$ in E_{B_n}. We obtain as an immediate consequence that

$$\begin{aligned} S\big(\sum(u_k)\big) &= \bigcup_n S_{B_n}\big(\sum(u_k)\big) \\ &\subset \bigcup_n S^e_{B_n}\big(\sum(u_k)\big) \qquad \text{(B1')} \\ &= \sum_{k=1}^{\infty} u_k + \bigcup_n A_{B_n}\big(\sum(u_k)\big) \\ &\subset \sum_{k=1}^{\infty} u_k + \bigcup_n Q_{B_n}\big(\sum(u_k)\big) \qquad \text{(B2')} \\ &= \sum_{k=1}^{\infty} u_k + \bigcup_n \Gamma^{\perp}_{B_n}\big(\sum(u_k)\big). \end{aligned}$$

Define

$$\Gamma^{\perp}_{\mathrm{loc}}\big(\sum(u_k)\big) := \bigcup_n \Gamma^{\perp}_{B_n}\big(\sum(u_k)\big)$$

and observe that this set is independent of the bounded set B and the fundamental system (B_n) we chose at the beginning. Since the sets $\Gamma^{\perp}_{B_n}$ form an increasing family of subspaces of E, the set $\Gamma^{\perp}_{\mathrm{loc}}$ is even a subspace of E.

THEOREM 2.1: *Let E be a complete nuclear (DF)-space.*

(a) *For each convergent series $\sum(u_k)$ in E the domain of sums $S\big(\sum(u_k)\big)$ is an affine subspace of E; more precisely:*

$$S\big(\sum(u_k)\big) = \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}_{\mathrm{loc}}\big(\sum(u_k)\big).$$

(b) *Assume E to be infinite dimensional and not isomorphic to φ. Then there is a convergent series $\sum(u_k)$ in E such that the domain of sums $S\big(\sum(u_k)\big)$*

is a nonclosed subspace of E; in particular:

$$S\Big(\sum(u_k)\Big) = \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}_{\mathrm{loc}}\Big(\sum(u_k)\Big) \subsetneq \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}\Big(\sum(u_k)\Big).$$

As in the metrizable case statement (a) will follow from the fact that the above inclusions (B1′) and (B2′) are even equalities. To prove this, it is necessary to adapt the lemma of rounding off coefficients and the permutation lemma (1.3 and 1.4 of section 1) to the (DF)-setting, and this is done in the following two lemmas:

LEMMA 2.2: *Let E be a vector space and $B_0 \subset B \subset C$ three Hilbert discs in E such that*

$$HS\big(i^B_{B_0}\colon E_{B_0} \hookrightarrow E_B\big) \le 1, \quad HS\big(i^C_B\colon E_B \hookrightarrow E_C\big) \le 1/2.$$

Then for each convergent series $\sum(u_k)$ in E_{B_0}

$$S^e_B\Big(\sum(u_k)\Big) \subset S_C\Big(\sum(u_k)\Big).$$

Proof: Take

$$x = \lim_{m\to\infty} \sum_{k=1}^{j_m} u_{\pi(k)} \in S^e_B\Big(\sum(u_k)\Big)$$

(convergence in E_B) and extract an increasing subsequence $(j_{m(\ell)})_{\ell\ge 2}$ of (j_m) such that for all $\ell \ge 2$

$$\begin{aligned} &x_{\pi(k)} \in 1/\ell B_0 \quad \text{for all } k = j_{m(\ell)}+1, \dots, j_{m(\ell+1)},\\ &x - \sum_{k=1}^{j_{m(\ell)}} u_{\pi(k)} \in 1/\ell B,\\ &x - \sum_{k=1}^{j_{m(\ell+1)}} u_{\pi(k)} \in 1/\ell B; \end{aligned}$$

this is possible: take first k_0 such that

$$u_{\pi(k)} \in 1/2B_0 \quad \text{for all } k \ge k_0,$$

and then m_0 such that $j_{m_0} \ge k_0$ and

$$x - \sum_{k=1}^{j_m} u_{\pi(k)} \in 1/2B \quad \text{for all } m \ge m_0.$$

Put $m(2) := m_0$. Then for $m \geq m(2)$

$$u_{\pi(k)} \in 1/2B_0 \quad \text{for all } k \geq j_{m(2)},$$
$$x - \sum_{k=1}^{j_m} u_{\pi(k)} \in 1/2B \quad \text{for all } m \geq m(2).$$

Select now $m(3) > m(2)$ such that

$$u_{\pi(k)} \in 1/3B_0 \quad \text{for all } k \geq j_{m(3)},$$
$$x - \sum_{k=1}^{j_m} u_{\pi(k)} \in 1/3B \quad \text{for all } m \geq m(3).$$

Then

$$u_{\pi(k)} \in 1/2B_0 \quad \text{for all } k = j_{m(2)} + 1, \ldots, j_{m(3)},$$
$$x - \sum_{k=1}^{j_{m(2)}} u_{\pi(k)} \in 1/2B,$$
$$x - \sum_{k=1}^{j_{m(3)}} u_{\pi(k)} \in 1/2B,$$

etc. .... Consider now the Hilbert discs

$$1/\ell B_0 \subset 1/\ell B \subset 1/\ell C.$$

Clearly,

$$HS\big(E_{1/\ell B_0} \hookrightarrow E_{1/\ell B}\big) \leq 1 \quad \text{and} \quad HS\big(E_{1/\ell B} \hookrightarrow E_{1/\ell C}\big) \leq 1/2.$$

Hence, by Lemma 1.4, for each $\ell \geq 2$ there exists a permutation σ_ℓ of $\pi\{j_{m(\ell)} + 1, \ldots, j_{m(\ell+1)}\}$ such that for all $m = j_{m(\ell)} + 1, \ldots, j_{m(\ell+1)}$

$$x - \sum_{k=1}^{j_{m(\ell)}} u_{\pi(k)} - \sum_{k=j_{m(\ell)}+1}^{m} u_{\sigma_\ell(\pi(k))} \in 1/\ell C.$$

Clearly, this gives a permutation ϱ of $\mathbb{N}$ such that (convergence in E_C)

$$x = \sum_{k=1}^{\infty} u_{\varrho(k)},$$

hence $x \in S_C\big(\sum(u_k)\big)$. ∎

LEMMA 2.3: *Let E be a vector space and $B_0 \subset B$ two Hilbert discs in E such that*

$$HS\Big(i_{B_0}^{B}\colon E_{B_0} \hookrightarrow E_B\Big) \leq 1.$$

Then for each convergent series $\sum(u_k)$ in E_{B_0}

$$A_B\big(\sum(u_k)\big) = Q_B\big(\sum(u_k)\big).$$

Proof: According to the definitions $A_B \subset Q_B$. Take $x \notin A_B$. This means, there is m_0 such that

$$x \notin \overline{Z_m}^{E_B} \quad \text{for all } m \geq m_0.$$

Choose $\varepsilon > 0$ such that

$$(x + \varepsilon B) \cap Z_{m_0} = \emptyset,$$

and $m_1 > m_0$ such that

$$u_k \in \varepsilon/2B_0 \quad \text{for all } k \geq m_1.$$

We will show that

$$(x + \varepsilon/2B) \cap co\, Z_{m_1} = \emptyset,$$

hence $x \notin Q_B$. If this is not the case, then there is

$$z \in (x + \varepsilon/2\, B) \cap co\, Z_{m_1}.$$

Since $HS\big(E_{\varepsilon/2\, B_o} \hookrightarrow E_{\varepsilon/2\, B}\big) \leq 1$, by Lemma 1.3 there is $\widetilde{z} \in Z_{m_1}$ with

$$z - \widetilde{z} \in \varepsilon/2B,$$

hence

$$\widetilde{z} \in \big(z + \varepsilon/2B\big) \cap Z_{m_1} \subset (x + \varepsilon\, B) \cap Z_{m_0},$$

a contradiction. ∎

Now we are prepared to give the

Proof of part (a) of Theorem 2.1: Let $\sum(u_k)$ be a convergent series in a complete nuclear (DF)-space E. Take a fundamental system (B_n) of $\mathcal{B}(E)$ such that $\sum(u_k)$ converges in E_{B_1}, all B_n are Hilbert discs and

$$HS\Big(i_{B_n}^{B_{n+1}}\colon E_{B_n} \hookrightarrow E_{B_{n+1}}\Big) \leq 1/2 \quad \text{for all } n$$

(see the preliminaries and combine e.g. [Ju, 7.8.2(4)] with [Ju, 7.6.3(3)]). Then the inclusion (B1′) by Lemma 2.2 and the inclusion (B2′) by Lemma 2.3 are equalities which gives the claim. ∎

This proof obviously allows collecting a bit more information about domains of sums in nuclear (DF)-spaces:

REMARK 2.4: *Let $\sum(u_k)$ be a convergent series in a complete nuclear (DF)-space E, and (B_n) a fundamental system of $\mathcal{B}(E)$ such that $\sum(u_k)$ converges in E_{B_1}. Then*

$$\begin{aligned} S\big(\textstyle\sum(u_k)\big) &= \bigcup_n S_{B_n} = \bigcup_n S^e_{B_n} \\ &= \sum_{k=1}^{\infty} u_k + \bigcup_n A_{B_n} = \sum_{k=1}^{\infty} u_k + \bigcup_n Q_{B_n} \\ &= \sum_{k=1}^{\infty} u_k + \bigcup_n \Gamma^{\perp}_{B_n} = \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}_{\mathrm{loc}}\big(\textstyle\sum(u_k)\big). \end{aligned}$$

The proof is clear from what was said before (note that all unions are independent of the fundamental system (B_n) which hence can be chosen as in the above proof of 2.1 (a)).

It remains to give a

Proof of part (b) of Theorem 2.1: The nuclear Fréchet space $F := E'_b$ is not isomorphic to $w := \prod_{\mathbb{N}} \mathbb{R}$ (recall that E is reflexive). Accordingly, we can apply [DV, Theorem 5] to conclude the existence of a quotient G of F which has a continuous norm $\|\cdot\|$, but is not countably normable (a projective limit with injective connecting mappings). We choose a fundamental system $(\|\cdot\|_n)$ of seminorms in G such that $\|\cdot\| \leq \|\cdot\|_1$ and all the completions of the normed spaces $(G, \|\cdot\|_n)$ are separable and reflexive (see e.g. [Ju, 7.6.3]). Since F and G are nuclear Fréchet spaces, it follows that $H = G'_b$ is a closed, topological subspace of $E = F'_b$. For each n define

$$B_n := \{x \in G \mid \|x\|_n \leq 1\}^0 \subset H(\text{polar in } H) \quad \text{and} \quad H_n := H_{B_n}.$$

We have $H := \operatorname{ind}_n H_n$, and since $\|\cdot\|_1$ is a norm on G, the Banach space H_1 is dense in H. As G is not countably normable, we can apply [BMT, 2.9 and 2.6] to conclude

$$\bigcup_n \overline{H}_1^{H_n} \subsetneq \overline{H}_1^{H} = H.$$

By [KK, Ex. 3.1.5], since H_1 is separable, there is a convergent series $\sum(u_k)$ in H_1 such that its domain of sums in H_1 coincides with H_1. The domain of sums $S\big(\sum(u_k)\big)$ in H is a subspace of H, which contains H_1 but is not closed; indeed,

if it is closed, it must coincide with H. On the other hand,

$$S\big(\sum(u_k)\big) \subset \bigcup_n S_{B_n}\big(\sum(u_k)\big) \subset \bigcup_n \overline{H}_1^{H_n} \neq H.$$

This is a contradiction. ∎

In view of the remark made in the introduction the following consequence of 2.1 (b) seems to be notable:

COROLLARY 2.5: *Let E be a complete and infinite dimensional, nuclear (DF)-space which is not isomorphic to φ. Then there is a convergent series $\sum(u_k)$ in E and some $x \in E$ such that for all $x' \in E'$ there is a permutation π of $\mathbb{N}$ with*

$$x'(x) = \sum_{k=1}^{\infty} x'\big(u_{\pi(k)}\big),$$

but $x \notin S\big(\sum(u_k)\big)$.

We finish this section showing that our approach to the Levy–Steinitz type theorems via local convergence and bounded sets covers a much larger class of spaces than only nuclear (DF)-spaces — it turns out that co-nuclearity and not nuclearity is the appropriate assumption needed.

Let E be a locally convex space in which every sequence converges locally, and let $\sum(u_k)$ be a series in E which converges in, say, E_{B_0}. Denote all absolutely convex and bounded supersets of B_0 by $\mathcal{B}$. Clearly,

$$\begin{aligned} S\big(\sum(u_k)\big) &= \bigcup_{\mathcal{B}} S_B \subset \bigcup_{\mathcal{B}} S_B^e \\ &= \sum_{k=1}^{\infty} u_k + \bigcup_{\mathcal{B}} A_B \\ &\subset \sum_{k=1}^{\infty} u_k + \bigcup_{\mathcal{B}} Q_B \\ &= \sum_{k=1}^{\infty} u_k + \bigcup_{\mathcal{B}} \Gamma_B^{\perp} =: \sum_{k=1}^{\infty} u_k + \Gamma_{\mathrm{loc}}^{\perp}\big(\sum(u_k)\big), \end{aligned}$$

where the notation of all new symbols is obvious; again the set $\Gamma_{\mathrm{loc}}^{\perp}$ is independent of the choice of B_0, and it can easily be shown that it is even a subspace of E. Another application of 2.2 and 2.3 gives the following extension of part (a) of Theorem 2.1:

THEOREM 2.6: *Let E be a complete, co-nuclear space for which each convergent sequence converges locally. Then the domain of sums of each convergent series $\sum(u_k)$ in E is an affine subspace,*

$$S\Big(\sum(u_k)\Big) = \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}_{\mathrm{loc}}\Big(\sum(u_k)\Big).$$

Before we give a list of new examples of spaces such that each domain of sums is affine, let us show that the preceding result even covers Banaszczyk's rearrangement theorem for metrizable spaces. For this it suffices to check the following remark, since a metrizable space is nuclear if and only if it is co-nuclear, and each convergent sequence in a metrizable space converges locally (see the preliminaries).

REMARK 2.7: *For each convergent series $\sum(u_k)$ in a metrizable, locally convex space E*

$$\Gamma^{\perp}\Big(\sum(u_k)\Big) = \Gamma^{\perp}_{\mathrm{loc}}\Big(\sum(u_k)\Big).$$

Proof: Clearly, by 1.1

$$\begin{aligned}\Gamma^{\perp}_{\mathrm{loc}} &= \bigcup_{B} \Gamma^{\perp}_{B} = \bigcup_{B} \bigcap_{m} \overline{co\, Z_m}^{E_B} \\ &\subset \bigcap_{m} \overline{co\, Z_m}^{E} = \Gamma^{\perp},\end{aligned}$$

hence take $x \in \bigcap_m \overline{co\, Z_m}^{E}$. If (U_n) is a decreasing basis of zero neighbourhoods in E, then for every m there is

$$x_m \in co Z_m \cap (x + U_m).$$

In particular, $x_n \longrightarrow x$ in E, and therefore $x_n \longrightarrow x$ in some E_C. Since $co\, Z_m \supset co\, Z_{m+1}$, the sequence $(x_n)_{n \geq m}$ is contained in $co\, Z_m$ for all m, hence $x \in \overline{co\, Z_m}^{E_C}$ for all m. ∎

As announced, we finally give a list of natural examples:

Examples 2.8:

1. A Köthe matrix $A = (a_n)_n$ is a sequence of scalar sequences satisfying $0 < a_n(i) \leq a_{n+1}(i)$ for each n, i. The Köthe echelon space associated with A is the Fréchet space defined by

$$\lambda_1(A) := \Big\{ x \in \mathbb{R}^{\mathbb{N}} \;\Big|\; p_n(x) := \sum_i a_n(i)|x(i)| \text{ for all } n \Big\},$$

endowed with the metrizable, locally convex topology generated by the fundamental system of seminorms (p_n). The space $\lambda_1(A)$ is nuclear if and only if for each n there is $m > n$ such that $\big(a_n(i)/a_m(i)\big)_i \in \ell_1$. If $\lambda_1(A)$ is nuclear, then its strong dual is the complete nuclear (DF)-space

$$\begin{aligned}\lambda_1(A)'_b &= k_\infty(A)\\ &= \Big\{u \in \mathbb{R}^{\mathbb{N}} \;\Big|\; \exists n : \|u\|_n := \sup_i |u(i)|/a_n(i) < \infty\Big\}.\end{aligned}$$

In fact, $\lambda_1(A)'_b = \operatorname{ind}_n \ell_\infty(1/a_n)$ is an (LB)-space and its topology can be described by the seminorms

$$p_v(u) := \sup_i v(i)|u(i)|$$

as v varies in the set of nonnegative sequences such that $\big(a_n(i)v(i)\big)_i \in \ell_\infty$ for all n. For more details and examples we refer to [Bi] and [MV]. One of the most important examples is the space s of rapidly decreasing sequences, which corresponds to $\lambda_1(A)$ for the Köthe matrix $(a_n(i)) := (i^n)$. The space of Schwartz tempered distributions S' on $\mathbb{R}^N$ is isomorphic to s'. Moreover, $C^\infty[0,1]$ and $\mathcal{D}(K)$, K a compact subset of $\mathbb{R}^N$ with nonempty interior, are isomorphic to s.

2. Let K be a nonvoid compact subset of $\mathbb{C}^N$. The space $H(K)$ of germs of holomorphic functions on K is a complete nuclear (DF)-space. Its topology is described as a countable inductive limit of Banach spaces in the following way: let (U_n) be a decreasing basis of open neighbourhoods of K satisfying $U_n \supset \overline{U_{n+1}}$. Let

$$i_n \colon H^\infty(U_n) \longrightarrow H^\infty(U_{n+1})$$

be the restriction map which is absolutely summing. Then

$$H(K) = \operatorname{ind}_n H^\infty(U_n).$$

3. An (LF)-space $E = \operatorname{ind}_n E_n$ is a Hausdorff countable, inductive limit of an increasing sequence of Fréchet spaces. An (LF)-space is called strict if every space E_n is a closed topological subspace of E_{n+1}. In this case, each E_n is a closed topological subspace of E, and every convergent sequence in E is contained and converges in a step E_n. Accordingly, if $E = \operatorname{ind}_n E_n$ is a strict (LF)-space and each step E_n is nuclear, we can apply Banaszczyk's original theorem to conclude that the domain of sums of each convergent series in E is given by the same formula. The most important example is the space of test functions for distributions $\mathcal{D}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^N$. By a result of Valdivia [Va] and Vogt

[V], $\mathcal{D}(\Omega)$ is in fact isomorphic to a countable direct sum $s^{(\mathbb{N})}$ of copies of the space s. We observe that the space $\mathcal{D}_{(\omega)}(\Omega)$ of test functions for ultradistributions of Beurling type on an open subset Ω of $\mathbb{R}^N$ is also a strict (LF)-space (see [BrMT]).

4. By the sequence space representation of Valdivia [Va] and of Vogt [V], the space of distributions $\mathcal{D}'(\Omega)$ on an open subset Ω of $\mathbb{R}^N$ is isomorphic to a countable product $(s')^{\mathbb{N}}$ of copies of the space s'. Therefore, it is a complete co-nuclear space. Moreover, by the permanence properties of the strict Mackey condition (see [BPC, 5.1.31]), every convergent sequence in $\mathcal{D}'(\Omega)$ converges locally, and our Theorem 2.6 can be applied to $\mathcal{D}'(\Omega)$. More spaces have a similar structure: the space of ultradistributions $\mathcal{D}'_{(\omega)}(\Omega)$ of Beurling type and the space $\mathcal{E}_{\{\omega\}}(\Omega)$ of ultradifferentiable functions of Roumieu type on an open subset Ω of $\mathbb{R}^N$ are also complete co-nuclear spaces which are the countable, projective limit of complete nuclear (DF)-spaces. In particular, they are subspaces of countable products of complete nuclear (DF)-spaces, and again the permanence properties of the strict Mackey condition ensure that Theorem 2.6 can be applied. We refer to [BrMT].

5. The space $A(\Omega)$ of real-analytic functions on an open subset Ω of $\mathbb{R}^N$ is endowed with a locally convex topology as follows: $A(\Omega) = \operatorname{proj}_K H(K)$, as K runs over all the compact subsets of Ω and $H(K)$ is defined as in 2. Deep results of Martineau from the 60's (cf. [BDom]) imply that $A(\Omega)$ is an ultrabornological, countable, projective limit of complete nuclear (DF)-spaces, and its dual $A(\Omega)'_b$ is a complete nuclear (LF)-space in which every convergent sequence is contained and converges in a step. Therefore, $A(\Omega)$ and $A(\Omega)'_b$ are both complete co-nuclear spaces to which Theorem 2.6 can be applied (again these spaces satisfy the strict Mackey condition by the argument given in 4.).

3. The converse

As already mentioned, Banaszczyk in [B2] proved the converse of his rearrangement theorem from [B1]. Modifying his cycle of ideas, we will obtain the following converse of our rearrangement theorem for nuclear (DF)-spaces from section 2 (Theorem 2.1(a)).

THEOREM 3.1: *Let E be a (DF)-space such that each convergent sequence converges locally. If for each convergent series $\sum(u_k)$ in E*

$$S\Big(\sum(u_k)\Big) = \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}_{\mathrm{loc}}\Big(\sum(u_k)\Big),$$

then E is nuclear.

We will first prove an appropriate characterization of nuclear (DF)-spaces which seems to be interesting in its own right. If X is an n-dimensional Banach space, λ_n the Lebesgue measure on $\mathbb{R}^n$ and ϕ: $\mathbb{R}^n \longrightarrow X$ a linear bijection, then

$$\mathrm{vol}_n(A) := \lambda_n\big(\phi^{-1}A\big), \quad A \text{ a Borel set in } X$$

gives a measure on the Borel sets of X. Although $\mathrm{vol}_n(A)$ changes with ϕ, the ratio $\mathrm{vol}_n(A)\ \mathrm{vol}_n(B)^{-1}$ for two Borel sets A and B is certainly independent. For a linear bijection T: $X \longrightarrow Y$ between two n-dimensional Banach spaces define

$$v_n(T) := \big(\mathrm{vol}_n(TB_X)/\,\mathrm{vol}_n(B_Y)\big)^{1/n};$$

for a linear operator T: $X \longrightarrow Y$ between two arbitrarily normed spaces set

$$v_n(T) := \sup v_n(T\colon M \longrightarrow TM),$$

where $n \in \mathbb{N}$ and the sup is taken with respect to all subspaces M of X such that $\dim M = \dim TM = n$; put $v_n(T) = 0$ whenever rank $T < n$. For this definition and the following (later very important) properties see [BB, Lemma 1] and also [Bl]:

(1) $v_n(T) \le \|T\|$.
(2) $v_n(TS) \le v_n(T)v_n(S)$.
(3) $v_n(RTS) \le \|R\|v_n(T)\|S\|$.
(4) $v_n(T) = \Big(\prod_{i=1}^{n} \delta_i(T)\Big)^{1/n}$, whenever T acts between Hilbert spaces and δ_i stands for the i-th Kolmogorov number.
(5) $v_n(T) \ge h_n(T)$, where $h_n(T)$ stands for the n-th Hilbert number.

Moreover, denote for each $0 < \varepsilon < 1$ the class of all operators T between normed spaces such that

$$\big(n^{\varepsilon} v_n(T)\big) \in \ell_\infty$$

by $\mathfrak{V}_\varepsilon$. Part (b) of the next proposition is implicit in [BB, Lemma 2].

PROPOSITION 3.2: *Let $0 < \varepsilon < 1$.*

(a) *Each composition of three 2-summing operators belongs to $\mathfrak{V}_\varepsilon$.*
(b) *The composition of $k \ge 5/\varepsilon$ operators in $\mathfrak{V}_\varepsilon$ is nuclear.*

Proof: (a) From Pietsch's factorization theorem we know that such a composition factorizes through a nuclear operator T: $H_1 \longrightarrow H_2$, H_1 and H_2 Hilbert spaces. Hence it suffices to check that T belongs to $\mathfrak{V}_\varepsilon$: it is well-known that $\big(\delta_k(T)\big) \in \ell_1$, hence the assertion is an immediate consequence of

$$n^{\varepsilon} v_n(T) \overset{(4)}{=} n^{\varepsilon}\Big(\prod_{1}^{n} \delta_i(T)\Big)^{1/n} \le n^{\varepsilon}\frac{1}{n}\sum_{i=1}^{n} \delta_i(T).$$

(b) For every composition $T = T_k \circ \cdots \circ T_1$ of $k \geq 5/\varepsilon$ operators in $\mathfrak{V}_\varepsilon$ we have

$$\sup_\ell \ell^5 v_\ell(T) \overset{(2)}{\leq} \sup_\ell \prod_{i=1}^{k} \ell^\varepsilon v_\ell(T_i) =: c < \infty.$$

On the other hand, for each n

$$\begin{aligned}\delta_n(T) &\leq \Big(\prod_{\ell=1}^{n} \delta_\ell(T)\Big)^{1/n} \\ &\leq n\Big(\prod_{\ell=1}^{n} h_\ell(T)\Big)^{1/n} \quad \text{(see the proof of [P1, 12.12.3])} \\ &\overset{(5)}{\leq} \Big(\prod_{\ell=1}^{n} n v_\ell(T)\Big)^{1/n}.\end{aligned}$$

Together we obtain

$$\delta_n(T) n^3 \leq \Big(\prod_{\ell=1}^{n} \frac{n^4}{\ell^5} \ell^5 v_\ell(T)\Big)^{1/n} \leq c\Big(\prod_{\ell=1}^{n} \frac{n^4}{\ell^5}\Big)^{1/n} \longrightarrow 0, \quad n \longrightarrow \infty,$$

which shows that $\big(\delta_n(T) n\big) \in \ell_1$, hence by [P2, 11.12.2] the sequence of approximation numbers $\big(a_n(T)\big) \in \ell_1$. It is well-known that this assures the nuclearity of T (see [P2, 18.6.3]). ∎

As an immediate consequence we obtain the following corollary.

COROLLARY 3.3: *Let E be a locally convex space. Then the following are equivalent:*

(1) *E is nuclear.*

(2) $\forall 0 < \varepsilon < 1 (\exists 0 < \varepsilon < 1) \forall p \in cs(E) \exists p \leq q \in cs(E) : \pi_q^p \in \mathfrak{V}_\varepsilon$.

Dually, the following equivalence holds:

(1′) *E is co-nuclear.*

(2′) $\forall 0 < \varepsilon < 1 (\exists 0 < \varepsilon < 1) \forall B \in \mathcal{B}(E) \exists B \subset C \in \mathcal{B}(E) : i_B^C \in \mathfrak{V}_\varepsilon$.

The implication (2) $\curvearrowright$ (1) was stated in [BB, Lemma 2]. For our purposes we need a characterization of nuclear (DF)-spaces via volume numbers which uses bounded sets and continuous seminorms simultaneously — based on ideas of [Ju, 6.4.2] we prove:

PROPOSITION 3.4: *For every (DF)-space E the following are equivalent:*

(1) *E is nuclear.*

(2) $\forall 0 < \varepsilon < 1 (\exists 0 < \varepsilon < 1) \forall B \in \mathcal{B}(E) \forall p \in cs(E) : \pi_p i_B \in \mathfrak{V}_\varepsilon$.

Proof: The proof of (1) $\curvearrowright$ (2) is easy: clearly, in every nuclear space E each mapping $\pi_p i_B$ can be written as a composition of three nuclear maps, hence $\pi_p i_B \in \mathfrak{V}_\varepsilon$ for each $0 < \varepsilon < 1$ (Lemma 3.2). Conversely, assume that (2) holds for some $0 < \varepsilon < 1$ and take a set $B \in \mathcal{B}(E)$. By Corollary 3.3 it suffices to show that $i_B^C \in \mathfrak{V}_\varepsilon$ for some $B \subset C \in \mathcal{B}(E)$. Take a fundamental system (B_n) of bounded sets such that

$$B_1 := B \quad \text{and} \quad 2B_n \subset B_{n+1} \quad \text{for all } n.$$

Assume that

$$i_n := i_{B_1}^{B_n} \notin \mathfrak{V}_\varepsilon \quad \text{for all } n > 1.$$

By definition, for every n there is a k_n-dimensional subspace M_n of E_B such that

$$\dim M_n = \dim i_n M_n = k_n > k_{n-1},$$
$$k_n^\varepsilon v_{k_n}(i_n \colon M_n \longrightarrow i_n M_n) \geq n.$$

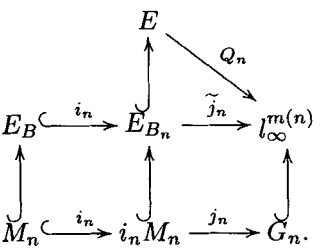
It is well-known (see e.g. [DF, 6.3]) that for each n there is a subspace G_n of some $\ell_\infty^{m(n)}$ and a linear bijection

$$j_n \colon i_n M_n \longrightarrow G_n,$$
$$\|j_n\| = 1,$$
$$\|j_n^{-1} \colon G_n \longrightarrow i_n M_n\| \leq 2.$$

By the Hahn–Banach theorem j_n has an extension $\widetilde{j}_n \colon E_{B_n} \longrightarrow \ell_\infty^{m(n)}$ with equal norm, and we know fom [Ju, 4.3.12] that there is an operator $Q_n \in \mathcal{L}(E, \ell_\infty^{m(n)})$ such that

$$\|Q_{n|E_{B_n}}\| \leq 2,$$
$$\|Q_n h - \widetilde{j}_n h\|_\infty \leq 1/4 \|h\|_{E_{B_n}} \quad \text{for all } h \in i_n M_n;$$

summarizing, we get the following diagram:



We prove that Q_n is invertible on $i_n M_n$ and

$$\|Q_n^{-1} : Q_n i_n M_n \longrightarrow i_n M_n\| \leq 4;$$

indeed, for $h \in i_n M_n$

$$\begin{aligned}\|h\|_{E_{B_n}} &= \|j_n^{-1} j_n h\|_{E_{B_n}} \leq 2\|j_n h\|_\infty \\ &\leq 2\|j_n h - Q_n h\|_\infty + 2\|Q_n h\|_\infty \leq 1/2\|h\|_{E_{B_n}} + 2\|Q_n h\|_\infty,\end{aligned}$$

hence

$$\|h\|_{E_{B_n}} \leq 4\|Q_n h\|_\infty.$$

The norm estimate for Q_n^{-1} can now be used to show that for each n

$$\begin{aligned}v_{k_n}\big(M_n \xrightarrow{i_n} i_n M_n\big) &= v_{k_n}\big(M_n \xrightarrow{i_n} i_n M_n \xrightarrow{j_n} G_n \xrightarrow{j_n^{-1}} i_n M_n\big) \\ &\leq 2v_{k_n}\big(M_n \xrightarrow{i_n} i_n M_n \xrightarrow{Q_n} Q_n i_n M_n \xrightarrow{Q_n^{-1}} i_n M_n \xrightarrow{j_n} G_n\big) \\ &\leq 8v_{k_n}\big(M_n \xrightarrow{i_n} i_n M_n \xrightarrow{Q_n} Q_n i_n M_n\big),\end{aligned}$$

hence

$$\begin{aligned}1/8n &\leq 1/8k_n^\varepsilon v_{k_n}\big(M_n \xrightarrow{i_n} i_n M_n\big) \\ &\leq k_n^\varepsilon v_{k_n}\big(M_n \xrightarrow{i_n} i_n M_n \xrightarrow{Q_n} Q_n i_n M_n\big),\end{aligned}$$

and finally

$$1/8n \leq k_n^\varepsilon v_{k_n}\big(E_B \overset{i_B}{\hookrightarrow} E \xrightarrow{Q_n} \ell_\infty^{m(n)}\big).$$

Define

$$V := \bigcap_{\mathbb{N}} Q_n^{-1} B_{\ell_\infty^{m(n)}} \subset E;$$

V absorbs bounded sets and is hence a zero neighbourhood in the (DF)-space E: clearly, for each k there is $\lambda > 0$ such that

$$B_k \subset \lambda \bigcap_{n=1}^{k} Q_n^{-1} B_{\ell_\infty^{m(n)}},$$

and for $n > k$ we have $2B_k \subset B_n$, hence

$$2Q_n B_k \subset Q_n B_n \subset 2B_{\ell_\infty^{m(n)}}.$$

Now observe that $Q_n V \subset B_{\ell_\infty^{m(n)}}$ for each n which assures that there are operators $\widehat{Q}_n$: $E_V \longrightarrow \ell_\infty^{m(n)}$ with $\|\widehat{Q}_n\| \leq 1$ and such that the following diagram commutes:

$$\begin{array}{ccccc} E_B & \xhookrightarrow{i_B} & E & \xrightarrow{Q_n} & l_\infty^{m(n)} \\ & & \downarrow{\scriptstyle \pi_V} & \nearrow{\scriptstyle \hat{Q}_n} & \\ & & E_V & & \end{array}$$

(as usual, $E_V = E_{m_V}$ and $\pi_V = \pi_{m_V}$). Since by assumption $\left(m^\varepsilon v_m(\pi_V i_B)\right) \in \ell_\infty$, we obtain a contradiction:

$$\begin{aligned} 1/8n &\le k_n^\varepsilon v_{k_n}(Q_n i_B) \\ &\le k_n^\varepsilon v_{k_n}\left(\widehat{Q}_n \pi_V i_B\right) \le k_n^\varepsilon v_{k_n}(\pi_V i_B). \end{aligned}$$ ∎

We need the following deep result of Banaszczyk [B2, Lemma 4] (which among others is based on Milman's quotient subspace theorem from local Banach space theory). Here $gp(A)$ denotes the subgroup generated by a subset A of a group G; moreover, for a seminorm p on a vector space E we write B_p for the closed unit ball with respect to p, and $d_p(x, H)$ for the distance of $x \in E$ and $H \subset E$ with respect to p.

THEOREM 3.5: *Let E be a vector space and $q \ge p$ two seminorms on E such that*

$$\left(\pi_q^p \colon E_q \longrightarrow E_p\right) \notin \mathfrak{V}_{0.1}.$$

Then for each finitely generated subgroup $G \subset E$, each $a \in \operatorname{span} G$ and each $\gamma > 0$ there is a finitely generated group $G_1 \supset G$ satisfying

$$\begin{aligned} &(1) \quad G_1 = gp(G_1 \cap B_q), \\ &(2) \quad d_p(a, G_1) \ge d_p(a, G) - \gamma. \end{aligned}$$

The following consequence will be crucial.

COROLLARY 3.6: *Let E be a locally convex space, $B \in \mathcal{B}(E)$ and $p \in cs(E)$ such that $\pi_p i_B \notin \mathfrak{V}_{0.1}$. Then there is a subgroup G of E such that*

$$\begin{aligned} &(1) \quad G = gp(G \cap 1/m\, B) \text{ for all } m, \\ &(2) \quad 1/2\, G \not\subset \overline{G}^p. \end{aligned}$$

Proof: Take $a \in E_B$ such that $p(a) > 0$ ($\pi_p i_B \neq 0$!). Without loss of generality we may assume that

$$p(a) = 2 \quad \text{and} \quad p \le \|\cdot\|_B \quad \text{on } E_B.$$

Define the finitely generated subgroup $G_0 := 2a\mathbb{Z}$ of E_B. Obviously,

$$p(a - 2a\, z) = 2|1 - 2z| \quad \text{for all } z \in \mathbb{Z},$$

hence $d_p(a, G_0) = 2$. Clearly, $(E_B)_p$ is an isometric subspace of E_p, hence the canonical map

$$\left(\pi_B^p \colon E_B \longrightarrow (E_B)_p\right) \notin \mathfrak{V}_{0.1}.$$

By Banaszczyk's Theorem 3.5 there is a finitely generated subgroup $G_0 \subset G_1 \subset E_B$ such that

$$G_1 = gp(G_1 \cap B),$$
$$d_p(a, G_1) \geq d_p(a, G_0) - 1/2.$$

Next apply Banaszczyk's result 3.5 to G_1, $2\|\cdot\|_B$, $2p$ and $1/2$: there is a finitely generated subgroup $G_2 \supset G_1$ such that

$$G_2 = gp\Big(G_2 \cap \frac{1}{2}B\Big),$$
$$d_{2p}(a, G_2) \geq d_{2p}(a, G_1) - \frac{1}{2}, \quad \text{hence} \quad d_p(a, G_2) \geq d_p(a, G_1) - \frac{1}{2^2}.$$

Proceeding this way, one gets an increasing sequence of finitely generated subgroups $G_n \supset G_{n-1}$ such that

$$G_n = gp\Big(G_n \cap \frac{1}{n}B\Big),$$
$$d_p(a, G_n) \geq d_p(a, G_{n-1}) - \frac{1}{2^n}.$$

Define the subgroup

$$G := \bigcup_n G_n$$

of E. Then for all m

$$G = gp\big(G \cap 1/mB\big);$$

indeed, for m and $x \in G$ we have $x \in G_n$ for some $n > m$, hence

$$x = \sum_{\text{finite}} g_i \quad \text{with } g_i \in G_n \cap 1/nB \subset G \cap 1/mB.$$

Moreover, $2a \in G_0 \subset G$, but

$$\begin{aligned} d_p(a, G) &= \inf\big\{d_p(a, G_k) \;\big|\; k \in \mathbb{N}\big\} \\ &\geq \inf\Big\{d_p(a, G_0) - \sum_{\ell=1}^{k} \frac{1}{2^\ell} \;\Big|\; k \in \mathbb{N}\Big\} \\ &= 2 - \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} = 1, \end{aligned}$$

hence $a \notin \overline{G}^p$. ∎

Finally, we are prepared to give a

Proof of Theorem 3.1: Assume that E is not nuclear. Then we know from Proposition 3.4 that

$$\pi_p i_B \notin \mathfrak{V}_{0.1}$$

for some $B \in \mathcal{B}(E)$ and $p \in cs(E)$. Hence by Corollary 3.6 there are a subgroup G of E and an $a \in E$ such that

(1) $G = gp(G \cap 1/mB)$ for all m,

(2) $2a \in G$, but $a \notin \overline{G}^p$.

Take a fundamental system (B_n) of $\mathcal{B}(E)$ such that $B_1 = B$. By (1) we have for each m the following finite representations of $2a$:

$$2a = \sum_{i=1}^{s(m)} w_i^m, \quad w_i^m \in G \cap 1/mB.$$

Define the series $\sum(u_k)$ by

$$(u_k) := \left(w_1^1, -w_1^1, \ldots, w_{s(1)}^1, -w_{s(1)}^1, w_1^2, -w_1^2, \ldots, w_{s(2)}^2, -w_{s(2)}^2, \ldots\right).$$

Obviously, the series $\sum(u_k)$ is convergent in E_{B_1} and we have (convergence in E_{B_1})

$$0 = \sum_{k=1}^{\infty} u_k \quad \text{and} \quad 2a \in A \subset \bigcup_n A_{B_n}.$$

We now show that $a \notin \bigcup_n A_{B_n}$ which then contradicts the fact that (by assumption and Remark 2.4)

$$S\left(\sum(u_k)\right) = \bigcup_n A_{B_n} = \Gamma_{\text{loc}}^{\perp}$$

is convex: assume that $a \in A_{B_n}$ for some n. Since for all m

$$Z_m \subset gp\left\{w_i^k \mid i = 1, \ldots, s(k) \quad \text{and} \quad k \in \mathbb{N}\right\} \subset G,$$

we get that

$$a \in A_{B_n} = \bigcap_m \overline{Z}_m^{E_{B_n}} \subset \overline{G}^{E_{B_n}} \subset \overline{G}^p,$$

a contradiction. ∎

An easy analysis of the preceding proof gives slightly more:

REMARK 3.7: *Let E be a locally convex space such that for some $B \in \mathcal{B}(E)$ and $p \in cs(E)$*

$$\pi_p i_B \notin \mathfrak{V}_{0.1}.$$

Then there is some E_B-convergent series $\sum(u_k)$ and there is $a \in E_B$ such that

$$2a \in \bigcap_m Z_m = A, \quad a \notin \bigcap_m \overline{Z}_m^p =: A_p.$$

This remark shows that our method also gives a new proof of the converse of Banaszczyk's rearrangement theorem for metrizable spaces; indeed, if E is a nonnuclear, metrizable space such that for each convergent series $\sum(u_k)$

$$S\big(\sum(u_k)\big) = \sum_{k=1}^{\infty} u_k + \Gamma^{\perp}\big(\sum(u_k)\big),$$

then by Lemma 1.2 for each such series

$$A_E\big(\sum(u_k)\big)$$

is a subspace. But this contradicts Remark 3.7, since $A_p \supset A_E \supset A$ and the following counterpart of Proposition 3.4 assures the existence of an appropriate B and p.

PROPOSITION 3.8: *For every metrizable space the statements* (1) *and* (2) *of Proposition* 3.4 *are equivalent.*

Proof: That (1) implies (2) follows as in Proposition 3.4. Hence assume that E satisfies (2) for some $0 < \varepsilon < 1$. Let $p \in cs(E)$; by Corollary 3.3 it suffices to show that there is some $p \le q \in cs(E)$ with $\pi_q^p \in \mathfrak{V}_\varepsilon$. Fix a fundamental system (p_n) of $cs(E)$ such that

$$p_n \ge p_1 = p \quad \text{for all } n,$$

and assume that

$$\pi_n := \pi_{p_n}^p \notin \mathfrak{V}_\varepsilon \quad \text{for all } n > 1.$$

By definition for every n there is a k_n-dimensional subspace M_n of E_{p_n} such that

$$\dim M_n = \dim \pi_n M_n = k_n > k_{n-1},$$
$$k_n^{\varepsilon} v_{k_n}(\pi_n : M_n \longrightarrow \pi_n M_n) \ge n.$$

The canonical embedding $i_n : M_n \hookrightarrow E_{p_n}$ being of finite rank has a finite representation

$$i_n = \sum \varphi_j' \otimes \pi_{p_j}(x_j) \in M_n' \otimes E_{p_n}.$$

Define the operator

$$R_n := \sum \varphi_j' \otimes x_j \colon M_n \longrightarrow E;$$

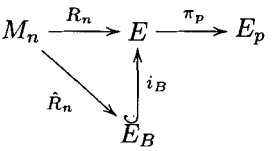
clearly, $i_n = \pi_{p_n} R_n$ for all n, hence

$$k_n^\varepsilon v_{k_n}(\pi_p R_n) = k_n^\varepsilon v_{k_n}(\pi_n \pi_{p_n} R_n)$$
$$= k_n^\varepsilon v_{k_n}(\pi_n i_n) = k_n^\varepsilon v_{k_n}(\pi_n \colon M_n \longrightarrow \pi_n M_n) \geq n.$$

Then the closed, absolutely convex hull B of $\bigcup_n R_n B_{M_n}$ is a bounded set in E; indeed, for k given, the set $\{x \in E \mid p_k(x) \leq 1\}$ absorbs the bounded set $\bigcup_1^k R_n B_{M_n}$, and for $n > k$

$$R_n B_{M_n} \subset \{x \in E \mid p_n(x) \leq 1\} \subset \{x \in E \mid p_k(x) \leq 1\}.$$

Observe now that $R_n B_{M_n} \subset B$ for all n, hence there are operators $\widehat{R}_n \colon M_n \longrightarrow E_B$ with norm ≤ 1 such that



commutes. But then

$$n \leq k_n^\varepsilon v_{k_n}(\pi_p R_n) = k_n^\varepsilon v_{k_n}(\pi_p i_B \widehat{R}_n) \leq k_n^\varepsilon v_{k_n}(\pi_p i_B)$$

contradicts the fact that $\pi_p i_B \in \mathfrak{V}_\varepsilon$. ∎

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